Recall Thm 10.7

(b) Every integrable $g(A)$-module V from the category $\mathcal{O}$ is isomorphic to a direct sum of modules $L(\lambda)$, $\lambda \in P_+$

Cor 10.7

(a). A $g(A)$-module $V \in \mathcal{O}$ is integrable

$\Longleftrightarrow$ $V = \bigoplus_{\lambda \in P_+} L(\lambda)$

(b). Tensor product of a finite number of integrable h.w.m module is a direct sum of $L(\lambda)$ with $\lambda \in P_+$

Rmk: Theorem 10.7 contains as a special case the classical weyl complete reducibility Thm

ϕ $L \longrightarrow gl(V)$, then ϕ is completly reducible

L: f.g. semisimple; $gl(V)$: f.d. repre

It follow thm 6.4 $\longrightarrow$

$V = \bigoplus_{\lambda \in h^*} V_\lambda$

$\dim V_\lambda < \infty$

$P(V) = D(\lambda_i) \cup \cdots$

$\Rightarrow V \in \mathcal{O}$

T_λ is integrable:

§10.8.

Let $s = (s_1, \dots s_n)$ be a sequence of integers in §1.5. $\rightsquigarrow$ $\mathbb{Z}$-gradation of type s:

$$\mathfrak{g}(A) = \bigoplus_{j \in \mathbb{Z}} \mathfrak{g}_j(s)$$

$$\mathfrak{g}_j(s) = \bigoplus_{\substack{\alpha \\ \alpha = \sum k_i \alpha_i \\ \sum_{i=1}^{n} k_i s_i = j}} \underset{\sim}{\mathfrak{g}_\alpha}$$

$$\underbrace{(k_1 \dots k_n)}_{\star} \longrightarrow \alpha \quad \star$$

$e_i \longrightarrow s_i$

A particular case of this the gradation of type $\mathbb{1} = (1 \dots 1)$ called the principal gradation.
Note that if $s_i > 0$, we have

$$\dim \mathfrak{g}_j(s) < \infty$$

$k_i \in \mathbb{Z}$

$\Big[$ $s_i > 0$ $\xrightarrow{\quad\quad\quad}$ $(s_1, \dots s_n) \begin{pmatrix} k_1 \\ \vdots \\ k_n \end{pmatrix} = j$

$\underset{\sim}{\dim \mathfrak{g}_\alpha} \le n^{|ht\,\alpha|}$ $\quad$ $\underline{s_j = 0}$

$k \to \infty$ choose

$\underline{s_i < 0}$ $\quad$ $\underline{s_j > 0}$ $\Big]$

Similarly:

$$\mathfrak{g}(^tA) = \bigoplus {}^t\mathfrak{g}_i(s)$$

$s = (s_1, \dots, s_n)$

Fix elements $\lambda_s \in \mathfrak{h}^*$ and $h^s \in \mathfrak{h}$ which satisfy

$\langle \lambda_s, \alpha_i^\vee \rangle = s_i$, $\langle h^s, \alpha_i \rangle = s_i$

$(i = 1 \dots n)$

Note that $\lambda_1 = \rho$ and $h^1 = \rho^\vee$

$\ulcorner$ $\rho \in \mathfrak{h}^*$ by equation $\langle \rho, \alpha_i^\vee \rangle = \frac{1}{2} a_{ii}$ $(i = 1 \dots n)$

$\det A = 0 \rightarrow$ A: GCM $a_{ii} = 1$

$\langle \rho^\vee, \alpha_i \rangle = \frac{1}{2} a_{ii}$ $(i = 1 \dots n)$ $\lrcorner$

Warning: $\nu : \mathfrak{h} \rightarrow \mathfrak{h}^*$

$\nu(\rho^\vee) \neq 2\rho / (\rho|\rho)$

$$\langle \rho, \alpha_i^\vee \rangle = 1 = \langle \rho^\vee, \alpha_i \rangle$$

$$(\nu^{-1}(\rho) | \alpha_i^\vee) \qquad (\rho^\vee | \nu^{-1}(\alpha_i)) = \frac{(\alpha_i|\alpha_i)}{2} (\rho^\vee | \alpha_i^\vee)$$

单纯

$(\cdot|\cdot)|_{\mathfrak{h}}$ is nondeg... $\Rightarrow \nu^{-1}(\rho) = \frac{(\alpha_i|\alpha_i)}{2} \rho^\vee$

$$v(e^{\nu}) = \frac{2}{(\alpha_i, \alpha_i)} e$$

固定

Provided that all $s_i > 0$, S

F_s : $\mathbb{C}[[e(-\alpha_1), \dots e(-\alpha_n)]] \longrightarrow \mathbb{C}[[q]]$ by

(10.8.1) $$F_s(e(-\alpha_i)) = q^{s_i} = \alpha_i(h^s) \quad (i = 1 \dots n)$$

This is called the specialization of type s, note that

(10.8.2) $$F_s(e(-\alpha)) = q^{\alpha(h^s)}$$

prop 10.8 $g(A) \to$ symme-- k_α- mod alg.

Then

$$\dim \tilde{g}_j(\mathbb{1}) = \dim^{+} g_j(\mathbb{1})$$

pf: Note that both side of identity (10.4.4)

$$\prod_{\alpha \in \Delta_+} (1 - e(-\alpha))^{\mathrm{mult}\,\alpha} = \sum_{w \in W} \varepsilon(w) e(w(\rho) - \rho)$$

are elements from the alge $\mathbb{C}[[e(-\alpha_1) \dots e(-\alpha_n)]]$

$\rho \in P_{++}$ $\quad \lambda - w(\lambda) = \sum c_i \alpha_i$

$\mathbb{C}$

Applying the homomorphism F_1 to both sides of (10.4.4)

$$F_1\left(\prod_{\alpha \in \Delta_+} (1 - e(-\alpha))^{\mathrm{mult}\,\alpha}\right) = \prod_{\alpha \in \Delta_+} \left(1 - q^{\alpha(h_1)}\right)^{\mathrm{mult}\,\alpha}$$

$$\dim \mathfrak{g}_j(\mathfrak{I}) = \dim \Big(\bigoplus_{\substack{\alpha \\ \alpha = \sum k_i \alpha_i \\ \sum k_i = j}} \mathfrak{g}_\alpha \Big)$$

$$\alpha(h_1) = \sum k_i \underbrace{\alpha_i(h_1)}_{=1} = j$$

$$= \prod_{j \geq 1} (1 - q^j)^{\dim \mathfrak{g}_j(\mathfrak{I})}$$

$$F_1\Big(\sum_{w \in W} \varepsilon(w)\, e(w(\rho) - \rho)\Big) = \sum_{w \in W} \varepsilon(w)\, q^{\langle \rho, \rho^\vee \rangle - \langle w(\rho), \rho^\vee \rangle}$$

$$\Rightarrow (10.8.3) \quad \prod_{j \geq 1} (1 - q^j)^{\dim \mathfrak{g}_j(\mathfrak{I})} = \sum_{w \in W} \varepsilon(w)\, q^{\langle \rho, \rho^\vee \rangle - \langle w(\rho), \rho^\vee \rangle}$$

Similary for $\mathfrak{g}({}^tA)$, we have

$$(10.8.4) \quad \prod_{j \geq 1} (1 - q^j)^{\dim {}^t\mathfrak{g}_j(\mathfrak{I})} = \sum_{w \in W({}^tA)} \varepsilon(w)\, q^{\langle \rho^\vee, \rho \rangle - \langle w(\rho^\vee), \rho \rangle}$$

$$W \longleftrightarrow W({}^tA)$$

$$\langle w(\rho), \rho^\vee \rangle = \langle \rho, w^*(\rho^\vee) \rangle$$

contragredient linear group!

$$\Rightarrow \dim \mathfrak{g}_j(\mathfrak{I}) = \dim {}^t\mathfrak{g}_j(\mathfrak{I}) \quad \#$$

Comparing (10.8.3) (10.8.4)

$$(10.8.5) \quad \prod_{\alpha \in \Delta_+} (1 - q^{\alpha(\rho^\vee)})^{\operatorname{mult} \alpha} = \prod_{\alpha^\vee \in \Delta^\vee_+} (1 - q^{\rho(\alpha^\vee)})^{\operatorname{mult} \alpha^\vee}$$

Rmk: F_s, some s_i are 0

F_s is not defined everywhere

$$F_s\left(e(-n\alpha_i)\right) = q^{\underbrace{n\alpha_i(h_s)}_{=s_i}} \to -\infty \quad (n\to\infty)$$

§ 10.9

The specialization of type 1 is called the pricipal specialization

give a product decomposition of pricipally specialization - character

Prop 10.9 $g(A)$, Let $\lambda \in P_+$,

$S = (\lambda(\alpha_1^\vee), \dots \lambda(\alpha_\ell^\vee))$.

$$F_{(1)}\left(e(-\lambda)\,ch L(\lambda)\right) = \prod_{j\geq 1}(1-q^j)^{\dim^+ g_j(s+1) - \dim^+ g_j(1)}$$

pf: By (10.4.5) $ch(L(\lambda)) = \dfrac{\sum_{w\in W}\varepsilon(w)\,e(w(\lambda+\rho))}{\sum_{w\in W}\varepsilon(w)\,e(w(\rho))}$

We have (10.9.2):

$$\underbrace{e(-\lambda)\,ch(L(\lambda))}_{F_1} = \frac{\sum_{w\in W}\varepsilon(w)\,e(w(\lambda+\rho)-(\lambda+\rho))}{\underbrace{\sum_{w\in W}\varepsilon(w)\,e(w(\rho)-\rho)}_{N_\lambda}}$$

For $\lambda \in P_{++}$, set $N_\lambda = \sum_{w \in W} \varepsilon(w)\, e(w(\lambda) - \lambda)$

Note $N_\lambda \in \mathbb{C}[[e(-\alpha_1), \dots e(-\alpha_n)]]$ by prop 3.12

(b) (d) i.e $\lambda \in C = \{\lambda \in \mathfrak{h}^* \mid \lambda(\alpha_i^\vee) \geq 0 \quad i = 1 \dots n\}$

$\lambda + \rho \in P_{++}$

(b) $\forall \lambda \in X$ $\quad W(\lambda) \cap C$ is exactly one point

(d) $C = \{\lambda \in \mathfrak{h}^* \mid \forall w \in W \ \underbrace{\lambda - w(\lambda)}_{} = \underbrace{\sum_i c_i \alpha_i}_{}$
$c_i \geq 0\}$

Let $\underbrace{r = (\lambda(\alpha_1^\vee), \dots \lambda(\alpha_n^\vee))}_{}$ $\quad h_1$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\|$

$F_1(N_\lambda) = \sum_{w \in W} \varepsilon(w)\, q^{\langle \lambda - w(\lambda), \rho^\vee \rangle}$

$= \sum_{w \in W} \varepsilon(w)\, q^{\lambda(\rho^\vee) - \underbrace{\langle w(\lambda), \rho^\vee \rangle}_{\|}}$

$= \sum_{w \in W} \varepsilon(w)\, q^{\langle \lambda, \rho^\vee \rangle - \langle \lambda, w(\rho^\vee) \rangle}$

$= \sum_{w \in W} \varepsilon(w)\, q^{\langle \lambda, \rho^\vee - w(\rho^\vee) \rangle}$

$= F_r\left(\sum_{w \in W} \varepsilon(w)\, e(w(\rho^\vee) - \rho^\vee)\right)$ ☆ $= F_r\left(\prod_{\alpha \in \Delta_+^\vee} (1 - e(-\alpha))^{\text{mult}\,\alpha}\right)$

$r = (\lambda_s(\alpha_1^\vee) \ \dots \ \lambda_s(\alpha_n^\vee))$

$\sum_{w \in W} \varepsilon(w)\, q^{\langle \rho^\vee - w(\rho^\vee), \lambda_s \rangle}$

(10.4.4) $\prod_{\alpha\in\Delta_+^\vee}(1-e(-\alpha^\vee))^{\mathrm{mult}\,\alpha^\vee} = \sum_{w\in W}\varepsilon(w)\,e(w(\rho^\vee)-\rho^\vee)$

for $\mathfrak{g}({}^tA)$

$F_1(N_\lambda) = F_1\left(\prod_{\alpha\in\Delta_+^\vee}(1-e(-\alpha))^{\mathrm{mult}\,\alpha}\right.$

$$\| $$

$$\prod_{\alpha\in\Delta_+^\vee}(1-q^{\lambda(\alpha)})^{\mathrm{mult}\,\alpha}$$

$\Rightarrow$ (10.9.3) $F_1(N_\lambda) = \prod_{\alpha\in\Delta_+^\vee}(1-q^{\lambda(\alpha)})^{\mathrm{mult}\,\alpha}$

Hence $F_1(e(-\Lambda)\,\mathrm{ch}\,L(\Lambda)) = F_1\left(\frac{N_{\Lambda+\rho}}{N_\rho}\right)$

$$= \left[\frac{\prod_{\alpha\in\Delta_+}(1-q^{\underbrace{\Lambda+\rho}(\alpha)})^{\underbrace{\mathrm{mult}\,\alpha}}}{\prod_{\alpha\in\Delta_+^\vee}(1-q^{\rho(\alpha)})^{\underbrace{\mathrm{mult}\,\alpha}}}\right] \star$$

(10.9.4)

$\Lambda(\alpha_i)=s_i \qquad \rho(\alpha_i)=1$

$\star$ $\dim\mathfrak{g}_j(s+1) = \dim\left(\bigoplus_{\substack{\alpha=\sum k_i\alpha_i \\ \sum k_i(s_i+1)=j}}\mathfrak{g}_\alpha\right) =: \;$

$\dim\mathfrak{g}_j(1) = \dim\left(\bigoplus_{\substack{\alpha=\sum k_i\alpha_i \\ \sum k_i=j}}\mathfrak{g}_\alpha\right) =: \;$

$\dim^t\mathfrak{g}_j(s+1) - \dim^t(\mathfrak{g}_j(1))$

$$= \prod_{j\geq 1}(1-q^j) \qquad \frac{\quad}{(/0.q.))}$$

$\dim g_j(1)$ $\quad \sum k_i(s_i+1)$ $\quad \sum k_i$

$$\prod_{\alpha\in\Delta_+^\vee}\left(1-q^{\wedge \cdot e(\alpha)}\right)^{\operatorname{mult}\alpha} = \prod_{j\geq 1}(1-q^j)^{\dim g_j^t(s+1)}$$

$\dim g_j^t(s+1) \longrightarrow g_\alpha \longrightarrow \alpha\in\Delta_+^\vee$

$\dim g_j^t(1) \qquad (1-q^j)$

$\alpha = \sum k_i\alpha_i$

$\sum k_i = j$

$$\prod_{\alpha\in\Delta_+^\vee}\left(1-e\,q^{e(\alpha)}\right)^{\operatorname{mult}\alpha} = \prod_{j\geq 1}(1-q^j)^{\dim g_j^t(1)}$$

$j = \sum k_i$

§10.10.

$$V = \bigoplus_{\lambda\leq\Lambda}\bar V_\lambda \qquad \text{hw. m } \Lambda$$

fix $s = (s_1 \cdots s_n)$

$$\deg(\lambda) = \deg(\Lambda - k_i\alpha_i) = \sum_i^n k_i s_i \in \mathbb{Z}$$

Then setting $V_j(s) = \bigoplus \bar V_\lambda$

α_i: $\deg(\alpha_i) = s_i$

$$V = \bigoplus_{j \in \mathbb{Z}_+} V_j(s)$$

$\dim V_j(s) < \infty$ if all $s_i > 0$ ☆

- $\mathbb{1} = (1 \cdots 1) \Rightarrow$ pricipal gradation

- $\dim V_j(s) < \infty$ ☆ $\quad s_i > 0 \quad j \geq 0$

$$\boxed{F_s(e(-\Lambda)\, \mathrm{ch}\, V) = \sum_{j \geq 0} \dim V_j(s) q^j}$$

$$F_s\left(\sum_{\lambda \leq \Lambda} \dim V_\lambda\, e(\lambda - \Lambda) \right)$$

$$= \sum_{\lambda \leq \Lambda} (\dim V_\lambda)\, q^{(\Lambda - \lambda)(h_s)}$$

$$\begin{cases} \Lambda - \lambda = \sum_{i=1}^{n} k_i \alpha_i \\ j = \sum_{i=1}^{n} k_i \alpha_i(h_s) = \sum_{i=1}^{n} k_i s_i \end{cases} \Rightarrow q^j$$

$$\dim(V_j(s)) = \sum \dim \left(\bigoplus_{\substack{\Lambda - \lambda = k_i \alpha_i \\ j = k_i s_i}} V_\lambda(s) \right)$$

$$= \sum_{j \geq 0} (\dim V_j(s))\, q^j$$

$q \to 1$ $\dim V$

$\|$

Let $S = I = (1 \cdots 1)$

$$F_1(e(-\Lambda)\,\mathrm{ch}(V)) = \sum_{j\geq 0} \dim V_j(1)\, q^j$$

the q-dimension of V $\to \dim_q V$

prop 10.10 $V = L(\Lambda)$

$$L(\Lambda) = \bigoplus_{j\geq 0} L_j(I)$$ prop 10.9

$\Lambda \in P_+$ $S = (\Lambda(\alpha_1^\vee) \cdots \Lambda(\alpha_n^\vee))$

$g(A)$ --- ka - - -

10.9.1 dim . . .

$$\dim_q(L(\Lambda)) = F_1(e(-\Lambda)\,\mathrm{ch} V) = \prod_{j\geq 1} (1-q^j) \ldots$$

$$= \prod_{\alpha \in \Delta_+^\vee} \left(\frac{1-q^{(\Lambda+\rho)(\alpha)}}{1-q^{(\rho,\alpha)}} \right)^{\text{mult}} \quad (10.93)$$

$\parallel$

$\dim(L(\Lambda))$ $q \to 1$

#

Coro: $A \to$ f. type. $g(A) \to$ simple f.d. Lie alge., $\Lambda \in P_+$, $L(\Lambda)$ is finite dimension

$$\dim L(\Lambda) = \lim_{q\to 1} \dim_q (L(\Lambda)) = \lim_{q\to 1} \prod_{\alpha\in\Delta_+^\vee} \left(\frac{1-q^{(\Lambda+\rho)(\alpha)}}{1-q^{\rho(\alpha)}} \right)$$

$$\lim_{x\to 1} \left(\frac{1-x^m}{1-x^n} \right) = \frac{m}{n}$$

$$= \frac{(\Lambda+\rho)(\alpha^\vee)}{\quad}$$